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BALL-COVERING PROPERTY OF BANACH SPACES

BY

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ABSTRACT

We consider the following question: For a Banach space X, how many closed balls not containing the origin can cover the sphere of the unit ball? This paper shows that: (1) If X is smooth and with $\dim X = n < \infty$, in particular, $X = R^n$, then the sphere can be covered by $n+1$ balls and $n+1$ is the smallest number of balls forming such a covering. (2) Let Λ be the set of all numbers $r > 0$ satisfying: the unit sphere of every Banach space X admitting a ball-covering consisting of countably many balls not containing the origin with radii at most r implies X is separable. Then the exact upper bound of Λ is 1 and it cannot be attained. (3) If X is a Gateaux differentiability space or a locally uniformly convex space, then the unit sphere admits such a countable ball-covering if and only if X^* is w^*-separable.

1. Introduction

For a finite dimensional (resp. separable infinite dimensional) Banach space, it is clear that the sphere S_X of the unit ball of X can always be covered by finitely (resp. countably) many closed balls not containing 0 with arbitrarily small radius, since S_X is compact (resp. separable). But there are still some natural and interesting questions that arise. For example:

PROBLEM 1: *If X is finite dimensional, is there a smallest number of balls not containing the origin whose union covers the unit sphere S_X of X?*

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PROBLEM 2: *Suppose that $r > 0$ satisfying: if every Banach space X admits a sequence of balls not containing the origin with radii at most r whose union covers the unit sphere S_X of X, then X is separable. Is there an exact upper bound for all such r?*

PROBLEM 3: *By the Separation Theorem, we can show that if S_X is covered by a sequence of balls not containing 0, then the dual X^* of X is w^*-separable. Is the converse true?*

This paper focuses our attention on the above questions and presents the following results:

For the first question (Problem 1), it is shown that if X is smooth and with $\dim X = n < \infty$, then the smallest number is $n + 1$. The answer to the second question (Problem 2) is the exact upper bound is 1 but it cannot be attained in general through proving that for every $0 < r < 1$, if the sphere S_X admits a ball-covering which consists of countably many balls with radii at most r then X is separable, and that l^∞ admits such a countable ball-covering with radii 1. For the last question (Problem 3), it is shown that if X is a Gateaux differentiability space (GDS) or X is locally uniformly convex, then S_X admits such a countable ball-covering if and only if X^* is w^*-separable.

Now, we recall some definitions. The letter X will always be a real Banach space, B_X its unit ball, and S_X the sphere of B_X. We denote by $B(x, r)$ the closed ball centered at x with radius r; if no confusion arises, $B(x, r)$ also denotes the open ball. For a set $A \subset X$, coA stands for the convex hull of A.

Definition 1.1: By a ball-covering of S_X, we always mean a family of closed balls not containing the origin whose union contains S_X. If a ball-covering consists of countably many balls, then we call the family a countable ball-covering. A ball-covering $\{B_\iota\}_{\iota\in\Lambda}$ is said to be symmetric, if $\{-B_\iota\}_{\iota\in\Lambda} = \{B_\iota\}_{\iota\in\Lambda}$. We say that a ball-covering has radius r if all balls from the covering have radii at most r.

Definition 1.2: (i) A continuous convex function defined on an open convex set D of a Banach space X is said to be Gateaux (resp. Fréchet) differentiable at $x \in D$ if there exists $x^* \in X^*$ such that for every $y \in X$,

$$\lim_{t\searrow 0}\left[\frac{f(x+ty)-f(x)}{t} - \langle x^*, y\rangle\right] = 0$$

$$\left(\text{resp. } \lim_{t\searrow 0}\sup_{y\in B_X}\left[\frac{f(x+ty)-f(x)}{t} - \langle x^*, y\rangle\right] = 0\right).$$

In this case, the corresponding x^* is called the Gateaux (resp. Fréchet) derivative of f at x.

(ii) A Banach space X is said to be a Gateaux differentiability space (resp. an Asplund space) if every continuous convex function defined on a nonempty open convex set D of X is Gateaux (resp. Fréchet) differentiable at each point of a dense (resp. dense G_δ-) subset of D. We say that X is Gateaux (Fréchet) smooth, if its norm is Gateaux (Fréchet) differentiable everywhere off the origin.

(iii) Let C be a nonempty bounded closed convex set of a Banach space X. A point $x \in C$ is said to be an exposed point (resp. a strongly exposed point) of C, if there exists $x^* \in X^*$ such that $\langle x^*, y\rangle < \langle x^*, x\rangle$ for all y in C with $y \neq x$ (resp. the slices $\{y \in C : \langle x^*, y\rangle > \langle x^*, x\rangle - \alpha\}_{\alpha>0}$ form a local base of C at x), and the corresponding functional x^* is called an exposing (resp. a strongly exposing) functional of C and exposing (resp. strongly exposing) C at x. If the convex set C is contained in X^*, then we define the w^*-exposed point (resp. w^*-strongly exposed point) of C analogously with the functional x^* in X instead of X^{**}.

PROPOSITION 1.3 ([Ph]): *A Banach space X is a Gateaux differentiability space if and only if every w^*-compact convex set in X^* is the w^*-closed convex hull of its w^*-exposed points.*

PROPOSITION 1.4 ([Fab, Th. 2.12]): *If X is a Gateaux differentiability space, then the w^*-compactness coincides with the w^*-sequential compactness in X^*.*

PROPOSITION 1.5 ([Ph]): *Suppose that p is a Minkowski functional defined on the space X. Then p is Gateaux (resp. Fréchet) differentiable at x and with the Gateaux (resp. Fréchet) derivative x^* if and only if x^* is a w^*-exposed (resp. w^*-strongly exposed) point of C^* and exposing (resp. strongly exposing) by x, where C^* is the polar of the level set $C \equiv \{y \in X : p(y) \leq 1\}$.*

2. The smallest ball number of a ball-covering in finite dimensional spaces

Recall a nonempty bounded set $A \subset X^*$ is said to be a norming set of X, if there exists $\alpha > 0$ such that $p(x) \equiv \sigma_A(x) \equiv \sup_{x \in A}\langle x^*, x\rangle \geq \alpha\|x\|$ for all $x \in X$. In this section, we first deal with symmetric ball-coverings.

LEMMA 2.1: *Suppose that $\dim X = n < \infty$. Then:*

(i) *B_{X^*} has at least $2n$ exposed points which forms a symmetric norming set of X.*

(ii) *B_X has exactly $2n$ exposed points if and only if X is isometric to l_1^n.*

Proof: (i) Since $\dim X = n, X$ is a Gateaux differentiability space. By Proposition 1.3, B_{X^*} is the closed convex hull of its exposed points. There must be n exposed points $\{x_1^*, x_2^*, \ldots, x_n^*\}$ in the sphere S_{X^*} which are linearly independent. The symmetry of B_{X^*} implies that all $\{\pm x_1^*, \pm x_2^*, \ldots, \pm x_n^*\}$ are exposed points of B_{X^*}. Therefore, the n-dimensional polyhedron $P \equiv \mathrm{co}\{\pm x_1^*, \pm x_2^*, \ldots, \pm x_n^*\}$ satisfies $0 \in \mathrm{int}\, P$. Thus, P is a norming set of X.

(ii) The sufficiency is trivial since l_1^n has exactly $2n$ exposed points $\{\pm e_1, \pm e_2, \ldots, \pm e_n\}$, where $\{e_i\}_{i=1}^n$ is the standard unit vector basis of l_1^n.

Necessity. Suppose B_X has exactly $2n$ exposed points $\{\pm x_1, \pm x_2, \ldots, \pm x_n\}$, where $\{x_1, \ldots, x_n\} \subset S_X$ are linearly independent. Then

$$B_X = \mathrm{co}\{\pm x_1, \ldots, \pm x_n\}.$$

Define $T: X \to l_1^n$ by

$$Tx = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

for $x = \sum_{i=1}^n \lambda_i x_i \in X$. This explains why the linear mapping T satisfies $TB_X = B_{l_1^n}$, and which in turn implies T is an isometry from X to l_1^n. ∎

THEOREM 2.2: *Suppose that X is an n-dimensional Banach space. Then:*

(i) *S_X has a symmetric ball-covering consisting of $2n$ balls.*

(ii) *Every symmetric ball-covering of S_X contains at least $2n$ balls.*

Proof: (i) By Lemma 2.1 (i), B_{X^*} contains at least $2n$ exposed points $\{\pm x_1^*, \pm x_2^*, \ldots, \pm x_n^*\}$ which forms a (symmetric) norming set of X. Thus, its support function p,

$$p(x) \equiv \max\{|\langle x_1^*, x\rangle|, \ldots, |\langle x_n^*, x\rangle|\}, \quad x \in X$$

defines an equivalent norm on X. Let $\alpha > 0$ be such that $p(x) \geq \alpha\|x\|$ for all $x \in X$. Due to Proposition 1.5, for every pair $\pm x_i^* (1 \leq i \leq n)$, there exist $\pm x_i \in S_X$ such that the derivatives of the norm at $\pm x_i, \|\pm x_i\|' = \pm x_i^*$. For each fixed $1 \leq i \leq n$, let $B_{i,m}^{\pm}$ be the open balls defined by

$$B_{i,m}^{\pm} = B(m \pm x_i, m - 1/m), \quad m = 1, 2, \ldots.$$

Clearly, $B_{i,m}^{\pm} \subset B_{i,m+1}^{\pm}$ for all $m \in \mathbf{N}$, and every $B_{i,m}^{\pm}$ has a positive distance $1/m$ from the origin. We claim $S_X \subset \bigcup\{B_{i,m}^{\pm} : 1 \leq i \leq n, m \in \mathbf{N}\}$. Given $y \in S_X$, there is $x^* \in \{\pm x_i^*\}_{i=1}^n$ such that $\langle x^*, y\rangle \geq \alpha\|y\| = \alpha > 0$. We can assume $x^* = x_j^*$ for some $1 \le j \le n$. Thus, there exist $\beta \ge \alpha$ and $h_j \in H_j \equiv \{x \in X : \langle x_j^*, x\rangle = 0\}$ such that

$$y = \beta x_j + h_j.$$

We want to show $y \in \bigcup_{m=1}^{\infty} B_{j,m}^{+}$. Otherwise, for every $m \in \mathbf{N}$,

$$m - 1/m \le \|mx_j - y\| = \|(m-\beta)x_j - h_j\|.$$

Thus

$$\begin{aligned}-1/m &\le \|(m-\beta)x_j - h_j\| - m\\ &= \|(m-\beta)x_j - h_j\| - m\|x_j\|\\ &= (m-\beta)\Big[\|x_j - \frac{1}{m-\beta}h_j\| - \|x_j\|\Big] - \beta\\ &= \frac{\|xj - th_j\| - \|x_j\|}{t} - \beta\end{aligned}$$

where $t = 1/(m-\beta)$. Letting $m \to \infty$, we observe

$$0 \le \|x_j\|'(h_j) - \beta = \langle x_j^*, h_j\rangle - \beta = -\beta < 0,$$

and this is a contradiction. Therefore, we have shown that

$$S_X \subset \bigcup\{B_{i,m}^{\pm} : 1 \le i \le n, m \in \mathbf{N}\}.$$

This explains why $\{B_{i,m}^{\pm}\}$ is an open covering of S_X. Compactness of S_X further says that there exists a sub-covering of S_X that consists of finitely many $B_{i,m}^{\pm}$, say $\{B_{i,j}^{\pm} : 1 \le i \le n, 1 \le j \le m\}$ for some $m \in \mathbf{N}$. Non-decreasing monotonicity of $\{B_{i,j}\}$ in j implies $S_X \subset \bigcup\{B_{i,m}^{\pm} : 1 \le i \le n\}$.

(ii) Suppose that $\{B_\iota\}_{\iota\in\Lambda}$ is a symmetric ball-covering of S_X. By the Separation Theorem, there exist Λ (the cardinal number of Λ) functionals $\{x_\iota^*\}_{\iota\in\Lambda}$ with $\{-x_\iota^*\}_{\iota\in\Lambda} = \{x_\iota^*\}_{\iota\in\Lambda} \subset S_{X^*}$ such that for every $x \in S$, there is $x_\alpha^* \in \{x_\iota^*\}_{\iota\in\Lambda}$ satisfying $\langle x_\alpha^*, x\rangle > 0$. Thus, the support function p of $\{x_\iota^*\}_{\iota\in\Lambda}$ defined by

$$p(x) = \sup_{\iota\in\Lambda}\langle x_\iota^*, x\rangle \quad \text{for all } x \in X$$

is an equivalent norm on X, the closed convex hull of $\{x_\iota^*\}_{\iota\in\Lambda}$ has nonempty interior, and $\{x_\iota^*\}_{\iota\in\Lambda}$ contains at least n independent elements, say $\{y_1^*, y_2^*, \ldots, y_n^*\}$. Symmetry of $\{x_\iota^*\}_{\iota\in\Lambda}$ implies that $\{\pm y_1^*, \pm y_2^*, \ldots, \pm y_n^*\} \subset \{x_\iota^*\}_{\iota\in\Lambda}$, which completes our proof. ∎

Next, we show that for the space X with $\dim X = n$, every ball-covering of S_X contains at least $(n+1)$ balls; and if X is smooth, then S_X always has a ball-covering consisting of $n+1$ balls, which is the following.

THEOREM 2.3: *Suppose that X is an n-dimensional space. Then:*

(i) *Every ball-covering of S_X contains at least $n+1$ balls.*

If, in addition, X is smooth, then

(ii) *S_X admits a ball-covering consisting of $n+1$ balls.*

Proof: (i) Suppose that $\{B_\iota\}_{\iota\in\Lambda}$ is a ball-covering of S_X. Again by the Separation Theorem, for each B_ι there exists $x_\iota^* \in S_{X^*}$ such that $\langle x_\iota^*, x\rangle > 0$ for every $x \in B_\iota$. Thus, $\{x_\iota^*\}_{\iota\in\Lambda}$ is a norming set of X, which in turn implies $\{x_\iota^*\}_{\iota\in\Lambda}$ contains $n+1$ affinely independent elements $\{y_0^*, y_1^*, \dots, y_n^*\}$. (Otherwise, $\{x_\iota^*\}_{\iota\in\Lambda}$ is contained in a hyperplane of X^*, and this contradicts that $\{x_\iota^*\}_{\iota\in\Lambda}$ is a norming set.) Therefore, $\{B_\iota\}_{\iota\in\Lambda}$ has at least $n+1$ elements.

(ii) Since $\dim X = n < \infty$ and since X is smooth, every point in S_{X^*} is an exposed point of B_{X^*}. We can choose $n+1$ points $\{x_0^*, x_1^*, \dots, x_n^*\}$ in S_{X^*} and $\{x_0, x_1, \dots, x_n\}$ in S_X satisfying

(a) $\{x_0^*, x_1^*, \dots, x_n^*\}$ are affinely independent;

(b) the interior of $\mathrm{co}\{x_0^*, \dots, x_n^*\}$ contains the origin;

(c) $\langle x_j^*, x_i\rangle = \delta_{ij}$ for all $1 \le i, j \le n$

and

(d) $\|x_i\|' = x_i^*$ for all $0 \le i \le n$.

Indeed, by the Auerbach Theorem (see, for instance, [LT, Prop. 1.c.3]), there exist n points $\{x_i\}_{j=1}^n$ in S_X and n points $\{x_j^*\}_{j=1}^n$ in S_{X^*} such that $\langle x_j^*, x_i\rangle = \delta_{ij}$.

Let

$$x_0^* = \alpha\Big(\sum_{j=1}^n (-x_j^*)\Big), \quad \text{where } \alpha = \frac{1}{\|\sum_{j=1}^n x_j^*\|}.$$

Then it is easy to show that $\{x_0^*, x_1^*, \dots, x_n^*\}$ and $\{x_1, \dots, x_n\}$ satisfy (a), (b) and (c). Since S_X is smooth, we have $\|x_i\|' = x_i^*$ for $i = 1, 2, \dots, n$. Choose $x_0 \in S_X$ such that $\langle x_0^*, x_0\rangle = 1$. Then $\|x_0\|' = x_0^*$. Thus $\{x_j^*\}_{j=0}^n$ and $\{x_i\}_{i=0}^n$ satisfy (d).

Because $\{x_0^*, x_1^*, \dots, x_n^*\}$ is a norming set of X, there exists $\beta > 0$ such that

$$\max\{\langle x_0^*, x\rangle, \dots, \langle x_n^*, x\rangle\} \ge \beta\|x\| \quad \text{for all } x \in X.$$

Thus,

$$S_X \subset \bigcup_{j=0}^n \{x \in S_X : \langle x_j^*, x\rangle \ge \beta\}.$$

Analogous to the proof of Theorem 2.2(i), we can show that there are $n+1$ balls $\{B_j\}_{j=0}^n$ not containing 0 such that

$$B_j \supset \{x \in S_X : \langle x_j^*, x\rangle \geq \beta\} \quad \text{for } j = 0, 1, \dots, n. \qquad \blacksquare$$

Remark 2.4: The proof of both Theorem 2.2(i) and 2.3(ii) closely depends on that there are points $\{x_i\}$ and $\{x_i^*\}$ with $\|x_i\|' = x_i^*$ for all i such that the convex hull $\mathrm{co}\{x_i^*\}$ forms a norming set of X. The following counterexample explains that these conditions are necessary to guarantee that the corresponding theorems hold.

Example 2.5: Let $X = l_1^2$ or l_∞^2. Then the dual unit balls $B_{l_\infty^2}$ and $B_{l_1^2}$, resp. have exactly 4 exposed points $(\pm 1, \pm 1)$ and $(\pm 1, 0), (0, \pm 1)$, resp. and the convex hull of each three of the four points does not contain the origin in its interior. This means that the convex hull of each three of the four points is not a norming set of X, and S_X can never be covered by three balls not containing the origin. Thus, the answer to the problem "whether the unit sphere of every n-dimensional Banach space admits a ball-covering consisting of $n+1$ balls" is negative in general.

3. Ball-covering properties of separable spaces

For every separable Banach space X and for $\varepsilon > 0$ there exists a countable ball-covering of S_X with the radii of the balls at most ε. On the other hand, if X is not separable, then for every $\varepsilon > 0$ there exists an uncountable net $\{x_\iota\} \subset S_X$ such that

$$\|u_\xi - x_\eta\| > 1 - \varepsilon \quad \text{for all } \xi \neq \eta$$

and this implies that if a Banach space X admits a countable ball-covering for S_X such that the radii of all the balls are at most $r < \frac{1}{2}$, then X is separable. So the following question arises. Is there an exact upper bound r_0 such that if a Banach space X admits a countable ball-covering, the radii of the balls at most $r < r_0$, then X is separable? In this section, we show $r_0 = 1$ and it cannot be attained.

THEOREM 3.1: *Suppose $0 < r < 1$. If S_X has a countable ball-covering with radii at most r, then X is separable.*

Before starting the proof, we state the following two facts.

FACT 3.2: If the unit sphere S_X of X has a countable ball-covering with radii at most ρ, then the sphere of a ball with radius $r > 0$ has a countable ball-covering with radii at most $\rho \cdot r$.

Proof: Let $\{B_n\}$ be a countable ball-covering of S_X, and let $B_n = B(x_n, \rho_n)$ with $\rho_n \leq \rho$. Suppose B^r is a ball in X with radius $r > 0$. Without loss of generality, we can assume $B^r = B(0, r)$. If $y \in B_n$ with $\|y\| = 1$, then $ry \in rB(x_n, \rho_n) = B(rx_n, r\rho_n)$, and yet, $B(rx_n, r\rho_n) \equiv B_n^r$ does not contain 0 if $B(x_n, \rho_n)$ does not. Thus $\{B_n^r\}$ is a countable ball-covering of the sphere of B^r and with radii $r\rho_n$ at most $r \cdot \rho$.

FACT 3.3: If the sphere S_X of X has a countable ball covering $\{B_n\}_{n=1}^{\infty}$ with radii at most r, then for every $\varepsilon > 0, B_X$ can be covered by a countable family of balls with radii at most $r + \varepsilon$.

Proof: Suppose that S_X has a countable ball-covering with radii at most r. Given $\varepsilon > 0$, let $\{r_n\} \subset (0, 1]$ be a sequence which is dense in $(0, 1]$. By Fact 3.2, the sphere of $B(r_n) \equiv r_n B_X$ can be covered by a sequence of balls, namely, $\{B_n^i\}_{i=1}^{\infty}$, where $B_n^i = B(x_n^i, r_n^i)$ with radius r_n^i at most $r \cdot r_n < r$. Thus, $\{B_n^i\}_{i,n \in \mathbf{N}}$ covers a dense subset of B_X, and further, $\{B_{n,\varepsilon}^i\}$ covers the whole unit ball B_X, where $B_{n,\varepsilon}^i = B(x_n^i, r + \varepsilon)$.

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: Suppose S_X has a countable ball-covering $\{B(x_n, r_n)\}$ with $r_n \leq r < 1$ for all n. Let

$$\varepsilon_0 = (1 - r)/2, \quad \varepsilon_m = 2^{-m}\varepsilon_0$$

and

$$s_m = r + \sum_{i=1}^{m} \varepsilon_i \Big(\leq r + \sum_{i=1}^{\infty} \varepsilon_i = r + \varepsilon_0 = 1 - \varepsilon_0 < 1 \Big)$$

for all $m \in \mathbf{N}$.

By Fact 3.3, there is a sequence of balls $\{B_{n_1}\}_{n_1=1}^{\infty}$ with radii at most $s_1 \equiv r + \varepsilon_1$, whose union covers B_X. Note for each $n_1 \in \mathbf{N}$ the sphere of B_{n_1} has a countable ball-covering $\{B_{n_1}^m\}_{m=1}^{\infty}$ with radii at most $r(r + \varepsilon_1)$. Fact 3.3 again implies that B_{n_1} can be covered by a sequence $\{B_{n_1,n_2}\}_{n_2=1}^{\infty}$ with radii at most $r(r + \varepsilon_1) + r\varepsilon_1 < (r + \varepsilon_1)^2 = s_1^2$. Clearly, $\{B_{n_1,n_2}\}_{n_1,n_2 \in \mathbf{N}}$ covers B_X.

Inductively, for each $k \geq 2$, we obtain countably many balls

$$\{B_{n_1,n_2,\dots,n_k} : n_i \in \mathbf{N}, i = 1, 2, \dots, k\}$$

with radii at most $(r + \sum_{i=1}^{k-1} \varepsilon_i)^k < (1 - \varepsilon_0)^k$, whose union covers B_X.

For each fixed $k \in \mathbf{N}$, let A_k be the set of all centers of the balls $B_{n_1,\ldots,n_k}$. We want to show $A = \bigcup_{k=1}^{\infty} A_k$ is dense in B_X. For every $\varepsilon > 0$, choose $k \in \mathbf{N}$ such that $(1-\varepsilon_0)^k < \varepsilon$. For any $x \in B_X$, there exists $(n_1, \ldots, n_k) \in N^k$ such that $x \in B_{n_1,\ldots,n_k}$. Therefore, $d(A_k, x) < (1-\varepsilon_0)^k < \varepsilon$. Thus, A is dense in B_X and X is separable. ∎

The following result shows that there exists a non-separable Banach space whose unit sphere has a countable ball-covering with radii 1. Combining this with Theorem 3.1, we obtain the exact upper bound of radii to guarantee separability of S_X with a countable ball-covering, and yet it follows from the following example that the exact upper bound cannot be attained.

Example 3.4: The unit sphere of l^∞ with its natural norm admits a countable ball-covering with radii 1.

Proof: Let $e_n = (\beta_j^{(n)}) \in l^\infty$ denote the unit vector with

$$\beta_j^{(n)} = \delta_{nj} = \begin{cases} 1, & j = n, \\ 0, & j \neq n. \end{cases}$$

For every $1 < q < 2$, we claim that the sequence of balls $\{B_n^{\pm}\}_{n=1}^{\infty}$ forms a countable ball-covering of the unit sphere of l^∞, where

$$B_n^{\pm} = B(q(\pm e_n), 1), \quad n = 1, 2, \ldots.$$

In fact, for each $y = (\beta_j) \in l^\infty$ with $\|y\| = 1$, there exists β_i with $|\beta_i| > q - 1$. We can assume $\beta_j > q - 1$. Thus $\|qe_i - y\| = \|(q - \beta_i)e_i - h_i\|$, where $h_i = (\alpha_j)$ with

$$\alpha_j = \begin{cases} \beta_j, & j \neq i, \\ 0, & j = i. \end{cases}$$

Note $\|h_i\| = \sup_{j \neq i} |\beta_j| \leq \|y\| = 1$ and $0 < q - \beta_i < 1$. We obtain $\|qe_i - y\| \leq 1$. Hence $y \in B(qe_i, 1)$, and this says what we claimed is true.

Remark 3.5: The differentiability hypothesis is also implicit in the proof of Example 3.4, since the norm $\|\cdot\|$ of l^∞ is Fréchet differentiable at $x = (\alpha_j) \in l^\infty$ if and only if there exists $\delta > 0$ and $i \in \mathbf{N}$ such that $|\alpha_i| \geq |\alpha_j| + \delta$ whenever $j \neq i$ (see, for instance, [DGZ], also [WCY]). Thus $\|\cdot\|$ is Fréchet differentiable at $\pm e_n$ for all n with the Fréchet derivatives $\|\pm e_n\|' = \pm e_n \in (l^\infty)^*$, and $qe_i - y = (q - \beta_i)e_i - h_i$ is just the "orthogonal decomposition" of the one dimensional space $\mathrm{R}e_i \subset l^\infty$ and the hyperplane through the origin determined by the functional $e_i \in (l^\infty)^*$.

4. Non-separable spaces with w^*-separable dual

Though l^∞ is non-separable and yet its unit sphere admits a countable ball-covering with radius 1, we observe that its dual $(l^\infty)^* = l^1 \oplus c_0^\perp$ is w^*-separable. More generally, we have

PROPOSITION 4.1: *If the sphere S_X of a Banach space X admits a countable ball-covering, then X^* is w^*-separable.*

Proof: It is an easy consequence of the Separation Theorem, since S_X can be covered by a sequence $\{B_n\}$ of closed balls that do not contain the origin. We obtain, by the Separation Theorem, a sequence $\{x_n^*\}$ in X^* such that

$$\langle x_n^*, x\rangle > 0 \quad \text{for all } x \in B_n;$$

$S_X \subset \bigcup_{n=1}^\infty B_n$ implies that $\{x_n^*\}$ separates points of X. Therefore, the span of $\{x_n^*\}$ is w^*-dense in X^*. ∎

The following theorems (Theorems 4.3 and 4.5) show that if X is a Gateaux differentiability space or a locally uniformly convex space, the converse of Proposition 4.1 is also true. We first need a lemma.

LEMMA 4.2: *If X is a Gateaux differentiability space and its dual is w^*-separable, then there is a sequence $\{x_n^*\}$ of w^*-exposed points of B_{X^*} such that*

$$\sup_n \langle x_n^*, x\rangle = \|x\| \quad \textit{for all } x \in X.$$

Proof: Since X is a Gateaux differentiability space, the closed unit ball B_{X^*} of X^* is w^*-sequentially compact (see [Fab, Th. 2.1.2]). Assume $\{z_n^*\}$ is a w^*-sequentially dense sequence of B_{X^*}.

Let $E \subset S_{X^*}$ be the set of all w^*-exposed points of B_{X^*}. Then the convex hull $\mathrm{co}(E)$ of E is w^*-dense in B_{X^*}, and further, $\mathrm{co}(E)$ is w^*-sequentially dense in B_{X^*}. Thus, for each z_n^* there exists $\{z_{n,k}^*\}_{k=1}^\infty \subset \mathrm{co}(E)$ such that $z_{n,k}^* \xrightarrow{w^*} z_n^*$ $(k \to \infty)$. Thus, $\{z_{n,k}^*\}_{n,k\in\mathbf{N}}$ is a w^*-dense countable subset of $\mathrm{co}(E)$. By definition of $\mathrm{co}(E)$, for each $z_{n,k}^*$, there exist $q(n,k) \in \mathbf{N}$, $y_{(n,k,i)}^* \in E, \lambda_i \geq 0$ for $i = 1, 2, \ldots, q(n,k)$ with $\sum_{i=1}^{q(n,k)} \lambda_i = 1$ such that

$$z_{n,k}^* = \sum_{i=1}^{q(n,k)} \lambda_i y_{(n,k,i)}^*.$$

Let $A(n,k) = \{y_{(n,k,i)}^*\}_{i=1}^{q(n,k)}$. Then $A = \bigcup_{n,k\in\mathbf{N}} A(n,k)$ is again a countable subset of E which we denote by $A = \{x_n^*\}$ such that

$$\begin{aligned}\|x\| \geq \sup_n \langle x_n^*, x\rangle &= \sup_{x^* \in A} \langle x^*, x\rangle \geq \sup_{n,k \in \mathbf{N}} \langle z_{n,k}^*, x\rangle \\ &= \sup_{x^* \in co(E)} \langle x^*, x\rangle = \sup_{x^* \in B_{X^*}} \langle x^*, x\rangle = \|x\|. \qquad \blacksquare\end{aligned}$$

THEOREM 4.3: *Suppose that X is a Gateaux differentiability space. Then the sphere S_X admits a countable ball-covering if and only if X^* is w^*-separable.*

Proof: It suffices to show sufficiency. Since X is a Gateaux differentiability space, and since X^* is w^*-separable, by Lemma 4.2, there exists a sequence $\{x_n^*\}$ of w^*-exposed points of B_{X^*} such that

$$\sup_n \langle x_n^*, x\rangle = \|x\| \quad \text{for all } x \in X.$$

Due to Proposition 1.5, there exists a sequence $\{x_n\} \subset S_X$ such that

$$\|x_n\|' = x_n^*, \quad n = 1, 2, \ldots,$$

where $\|x_n\|'$ denote the Gateaux derivatives of $\|\cdot\|$ at x_n for all $n \in \mathbf{N}$.

Through an argument which is analogous to the proof of Theorem 2.2(i), we obtain

$$S_X \subset \bigcup_{m,n \in \mathbf{N}} B_{m,n}$$

where $B_{m,n} = B(mx_n, m - 1/m)$ for $m, n \in \mathbf{N}$. $\blacksquare$

COROLLARY 4.4: *If a Banach space X admits an equivalent Gateaux smooth norm, then S_X has a countable ball-covering if and only if X^* is w^*-separable.*

Proof: It suffices to note every Banach space admitting an equivalent Gateaux smooth norm is a Gateaux differentiability space ([Ph]). $\blacksquare$

Recall that a Banach space is said to be locally uniformly convex if for every $x \in S_X$ and $\{x_n\} \subset S_X$, $\|x_n + x\| \to 2$ implies $x_n \to x$.

THEOREM 4.5: *Suppose X is a locally uniformly convex Banach space. Then S_X has a countable ball-covering if and only if X^* is w^*-separable.*

Proof: Let $\{x_n^*\}_{n=1}^\infty \subset S_{X^*}$ be a w^*-dense subset of S_{X^*}. Since norm-attaining functionals on X are always dense in the dual X^* of X, we can assume for every $n \in \mathbf{N}$ that there exists $x_n \in S_X$ such that $\langle x_n^*, x_n\rangle = 1$. Let $B_{m,n} = B(mx_n, m - 1/m)$ for all $m, n \in \mathbf{N}$.

If there exists y with $\|y\| = 1$ such that

$$y \notin B_{m,n} \quad \text{for all } m, n \in \mathbf{N},$$

or equivalently, $m - 1/m < \|mx_n - y\|$ for all $m, n \in \mathbf{N}$, w^*-density of $\{x_n^*\}$ in S_X^* implies that for every $\varepsilon > 0$, there exists x_n^* such that

$$\langle x_n^*, y\rangle > 1 - \varepsilon.$$

In particular, letting $\varepsilon = 1/k$ for $k = 1, 2, \dots$ we obtain a subsequence $\{x_{n_k}^*\}$ of $\{x_n^*\}$ such that

$$\langle x_{n_k}^*, y\rangle > 1 - 1/k.$$

This implies

$$2 \geq \|x_{n_k} + y\| \geq \langle x_{n_k}^*, x_{n_k} + y\rangle > 2 - 1/k.$$

Locally uniform convexity of X in turn implies $x_{n_k} \to y$ as $k \to \infty$, thus $\|x_{n_k} - y/2\| \to \frac{1}{2}(k \to \infty)$. This is impossible since

$$\frac{y}{2} \notin \frac{1}{2}B_{2,n_k} = \frac{1}{2}B\Big(2x_{n_k}, 2 - \frac{1}{2}\Big) = B\Big(x_{n_k}, \frac{3}{4}\Big)$$

for all $k \in \mathbf{N}$. ∎

Though we do not know whether w^*-separability of X^* can imply that the sphere S_X has a countable ball-covering, we can show S_X admits a countable "ball-cut-like set" covering.

Definition 4.6: (i) A set A in a Banach space X is said to be a ball-cut, if there exist an affine subspace M of X and a ball B in X such that $A = M \cap B$.

(ii) A is said to be ball-cut-like, if it is isometric to a ball-cut.

THEOREM 4.7: *If X^* is w^*-separable, then the sphere S_X of X admits a countable ball-cut-like set covering with diameters at most 2.*

Proof: It suffices to note Example 3.4 and note that every Banach space with a w^*-separable dual is isometric to a closed subspace of l^∞.

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